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Applications of the Polynomial Pair $f(s)$ and $f(S/\beta - \alpha)$ to Control, Probability, and Matrix Analysis

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Abstract: Let $f(s) := \Sigma_{i=0}^{n} a_i s^i$ be a monic polynomial (*i.e.*, $a_n = 1$) and let $F(S) := f(S/\beta - \alpha) = \Sigma_{i=0}^{n} b_i S^i$, $\beta \neq 0$. If the s_k's are the n roots of $f(s)$ then the $S_k = (s_k + \alpha)\beta$'s are the n roots of $F(S)$. *i.e.*, we obtain the roots of $F(S)$ by shifting the roots of $f(s)$ by α and then multiplying them by β. Let $\mathbf{P}(\alpha, \beta)$ denote the upper triangular $(n + 1)$-square matrix that transforms the coefficients of $f(s)$ to the coefficients of $F(S)$, let $\mathbf{P}(\alpha) := \mathbf{P}(\alpha, 1)$, and let $\mathbf{P} := \mathbf{P}(-1)$. We call the upper triangular matrix $\mathbf{P}$ Pascal's matrix since its nonzero elements coincide with Pascal's triangle. We will show the following results.

(i) If $\beta = 1$ and $f(s)$ is Hurwitz (*i.e.*, stable $\Re s_k < 0$, $k = 1, \ldots, n$) and $\alpha = a_{n-1}/n$, then $F(S)$ becomes a depressed polynomial (*i.e.*, $b_{n-1} = 0$) and hence unstable. We will show that $\sigma_u := a_{n-1}/n$ is an upper bound on $f(s)$'s stability margin, say σ. The application of the proposed upper bound is to narrow the search region of $f(s)$'s stability margin from $(0, \infty)$ to $(0, \sigma_u)$.

(ii) Waring's formulas that appear in probability state that $\mathbf{P}^{-1} = \mathbf{P}(1)$. By shifting the roots of $f(s)$ by α and then shifting the roots of $F(S)$ by $-\alpha$ we arrive back at $f(s)$. Hence, we obtain that $\mathbf{P}(-\alpha)\mathbf{P}(\alpha) = \mathbf{I}$, where $\mathbf{I}$ denotes the $(n+1)$-square identity matrix and hence $\mathbf{P}^{-1}(\alpha) = \mathbf{P}(-\alpha)$ thus generalizing Waring's formulas. Similarly, we will derive other matrix identities where multiplication is commutative. We will also show that the set of matrices $\{\mathbf{P}(\alpha) \,.\, \alpha \in \mathbb{R}\}$ form a multiplicative abelian group with identity matrix $P(0)$.

(iii) We obtained a more general formula by first substituting in $f(s)$, $s = S/\beta - \alpha$ and then substituting in $F(S)$, $S = \beta(s + \alpha)$ thus arriving back at $f(s)$. Hence, $\mathbf{P}^{-1}(\alpha, \beta) = \mathbf{P}(-\alpha\beta, \beta^{-1})$. We will also show that $\mathbf{P}(\alpha, \beta) = \mathbf{D}(\beta)\mathbf{P}(\alpha)$, where $\mathbf{D}(\beta)$ is a diagonal matrix and $\mathbf{D}_{ii}(\beta) = \beta^{-i}$, $i = 0, 1, \ldots, n$. Therefore, $\mathbf{P}^{-1}(\alpha, \beta) = \mathbf{P}(\alpha)^{-1}\mathbf{D}^{-1}(\beta) = \mathbf{P}(-\alpha)\mathbf{D}(\beta^{-1})$.

(iv) Finally, we will extend the results to multivariable polynomials.

Index Terms: Hurwitz polynomial, Depressed polynomial, Stability margin, Waring's formulas, Multiplicative abelian group, Kronecker product.

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I. Introduction

If $f(s)$ is the monic characteristic polynomial of a stable linear continuous system then $f(s)$ is called Hurwitz. Let s_k, $k = 1, \ldots, n$ denote the roots of $f(s) = a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$, $a_n = 1$. Then, $f(s) = \Pi_{k=1}^{n}(s - s_k)$ is Hurwitz iff

$$\Re s_k < 0,\ k = 1, \ldots, n. \tag{1}$$

The Routh-Hurwitz test detects whether or not a polynomial is Hurwitz, see (Dorf and Bishop, 2005, Chapter 6). Assuming that $f(s)$ is Hurwitz, for system design purposes we wish to find its stability margin, *i.e.*,

$$\sigma := \min(-\Re s_1, \ldots, -\Re s_n). \tag{2}$$

We can compute σ as follows

$$\sigma = \min\{\alpha : \alpha > 0,\ F(S) := f(S - \alpha),\ F(S) \text{ is not Hurwitz}\} \tag{3}$$

In Section II we will show that a simple upper bound on σ, say σ_u, coincides with $\alpha = a_{n-1}/n$ for which $F(S) = f(S - \alpha)$ becomes a depressed polynomial. $F(S)$ is called a depressed polynomial if the coefficient b_{n-1} of S^{n-1} is zero. Notice that depressed polynomials appear as an intermediate step in the derivation of explicit expressions for the roots of polynomials of degree 2-4. We will also comment on finding lower bounds on σ, say σ_ℓ. Initially, the article ended after Section II. However, when studying the transformation matrix $\mathbf{P}(\alpha)$ that transforms the coefficients of $f(s)$ to the coefficients of $f(S - \alpha)$ Pascal's matrix, $\mathbf{P}$, that appeared many times in (Hertz, 2021) and (Hertz, 2022) reappeared here, thus leading to the additional results in Sections III and IV.

In Section III we will derive explicit expressions for $\mathbf{P}(\alpha, \beta)$ and its inverse. In particular, when $\beta = 1$ we immediately obtain that $\mathbf{P}(\alpha)\mathbf{P}(-\alpha) = \mathbf{I}$ which we call the generalized Waring's formulas. Waring's formulas, $\mathbf{P}^{-1} = \mathbf{P}(1)$, correspond to $\alpha = 1$ and appear in probability theory, see (Grimmet and Stirzaker, 2001), (Hertz, 2021), and (Hertz, 2022). In (Hertz, 2021) and (Hertz, 2022). Pascal's matrix, $\mathbf{P} := \mathbf{P}(-1)$, appeared in many results and also in another more intuitive proof of Waring's formulas. The proof here of the generalized Waring's formulas is simpler than the latter two proofs and also extends Waring's formulas. We will also show that the matrices in $\{\mathbf{P}(\alpha) : \alpha \in \mathbb{R}\}$ are commutative under multiplication and form a multiplicative abelian group with identity matrix $\mathbf{P}(0)$. In the general case by first substituting in $f(s)$, $s = S/\beta - \alpha$ and then substituting in $F(S)$, $S = \beta(s + \alpha)$ we arrive back at $f(s)$. Hence, we obtain that $\mathbf{P}^{-1}(\alpha, \beta) = \mathbf{P}(-\alpha\beta, \beta^{-1})$. We will also show that $\mathbf{P}(\alpha, \beta) = \mathbf{D}(\alpha)\mathbf{P}(\alpha)$, where $\mathbf{D}(\alpha)$ is a diagonal matrix and $\mathbf{D}_{ii}(\beta) = \beta^{-i}$, $i = 0, 1, \ldots, n$. Therefore, $\mathbf{P}^{-1}(\alpha, \beta) = \mathbf{P}^{-1}(\alpha)\mathbf{D}^{-1}(\beta) = \mathbf{P}(-\alpha)\mathbf{D}(\beta^{-1})$. Notice that multiplication of matrices $\mathbf{P}(\alpha_i, \beta_i)$, $\beta_i \neq 1$, $i = 1, 2$ is not necessarily commutative. In Section IV we extend the results of Section III to multivariable polynomials. Finally, in Section V we give the conclusion.

II. Bounds on the Stability Margin of Hurwitz Polynomials

Let $\sigma_\ell > 0$ denote a lower bound on the stability margin of the Hurwitz polynomial $f(s)$. If $f(s)$ turns out to be stable, *e.g.*, by using the Routh Hurwitz test (Dorf and Bishop, 2005) then $\sigma_\ell > 0$. If for some $\alpha_0 > 0$, $f(S - \alpha_0)$ turns out to be Hurwitz then $\sigma_\ell = \alpha_0$.

Let $s = x + iy$, $(x, y) \in \mathbb{R}^2$. If by using the test in (Zeheb and Hertz, 1982) $f(s)$ turns out to be stable with respect to the left branch of the hyperbola $x^2/a^2 - y^2/b^2 = 1$, then obviously $\sigma_\ell = a$. Specifically, if by applying the Routh-Hurwitz test to the polynomial

$$g(z) := (z^2+1)^n f\left(\frac{az^2+2bz-a}{z^2+1}\right) \tag{4}$$

we obtain that if $g(z)$ is Hurwitz then $f(s)$ is stable with respect to the left branch of the above hyperbola and $\sigma_\ell = a$. If $f(s)$ is Hurwitz (*i.e.*, $\Re s_k < 0$, $k = 1, \ldots, n$) and $\alpha = a_{n-1}/n$, then $F(S) = f(S - \alpha)$ becomes a depressed polynomial (*i.e.*, $b_{n-1} = 0$ and hence unstable). We will show that $\sigma_u := a_{n-1}/n$ is an upper bound of $f(s)$'s stability margin, say σ. The application of the proposed upper bound is to narrow the search region of $f(s)$'s stability margin from (σ_ℓ, ∞) to (σ_ℓ, σ_u).

By using Vieta's formula $F(S)$ becomes a depressed polynomial if

$$\begin{aligned} b_{n-1} &= -(S_1 + \ldots + S_n) \\ &= -((s_1+\alpha) + \ldots + (s_n+\alpha)) \\ &= a_{n-1} - n\alpha = 0. \end{aligned} \tag{5}$$

Let $\alpha_d := a_{n-1}=n$ be the α that renders $F(S)$ a depressed polynomial. It is well known that $a_i > 0$, $i = 0, \ldots, n$ is a necessary condition for $f(s)$ to be Hurwitz (this result can be easily obtained by using $f(s) = \Pi_{i=1}^n (s - s_i)$ and $\Re s_i < 0 \ \forall i$). Notice that for $n \leq 2$ this is also a sufficient condition. Hence, since $F(S) = f(S - \alpha_d)$ is a depressed polynomial, $F(S)$ cannot be Hurwitz and therefore $\sigma < \sigma_u := a_{n-1}/n$. So instead of (3) we have

$$\sigma = \min\{\alpha : \sigma_\ell < \alpha < \sigma_u, F(S) := f(S-\alpha) \text{ is not Hurwitz}\}. \tag{6}$$

Remark 2.1. Notice that when $\Re s_i = -\sigma$, $i = 1, \ldots, n$, σ_u corresponding to the depressed polynomial $f(S - \sigma_u)$ becomes the stability margin of $f(s)$.

III. Properties of the Transformation Matrices P(α, β) and A Generalization of Waring's Formulas

Let $\mathbf{a} := (a_0, a_1, \ldots, a_n)^T$ denote the coefficients of $f(s)$ and $\mathbf{b} := (b_0, b_1, \ldots, b_n)^T$ denote the coefficients of $F(S) := f(S/\beta - \alpha)$. Then,

$$\begin{aligned} F(S) &= \sum_{i=0}^{n} a_i (S/\beta - \alpha)^i \\ &= \sum_{i=0}^{n} a_i \sum_{k=0}^{i} \binom{i}{k} S^k \beta^{-k} (-\alpha)^{i-k} \\ &= \sum_{i=0}^{n} a_i \sum_{k=0}^{n} \binom{i}{k} S^k \beta^{-k} (-\alpha)^{i-k}, \text{ since } \binom{i}{k} = 0 \text{ for } k > i \\ &= \sum_{k=0}^{n} S^k \beta^{-k} \sum_{i=0}^{n} a_i \binom{i}{k} (-\alpha)^{i-k} \end{aligned} \tag{7}$$

$$= \sum_{k=0}^{n} S^k \beta^{-k} \sum_{i=k}^{n} a_i \binom{i}{k} (-\alpha)^{i-k}$$

But

$$F(S) = \sum_{k=0}^{n} b_k S^k. \tag{8}$$

Hence,

$$b_k = \beta^{-k} \sum_{i=k}^{n} a_i \binom{i}{k} (-\alpha)^{1-k}, \; k = 0, 1, ..., n. \tag{9}$$

Therefore,

$$\mathbf{b} = \mathbf{P}(\alpha, \beta)\mathbf{a} = \mathbf{D}(\beta)\mathbf{P}(\alpha)\mathbf{a}, \tag{10}$$

where $\mathbf{D}(\beta)$ is the diagonal matrix

$$\mathbf{D}(\beta) := \mathrm{diag}(\beta^0, \beta^{-1}, ..., \beta^{-n}) \tag{11}$$

and

$$\mathbf{P}(-\alpha) = \begin{pmatrix} \alpha^0\binom{0}{0} & \alpha^{-1}\binom{1}{0} & \alpha^{-2}\binom{2}{0} & \alpha^{-3}\binom{3}{0} & \cdots & \alpha^{-n}\binom{n}{0} \\ 0 & \alpha^0\binom{1}{1} & \alpha^{-1}\binom{2}{1} & \alpha^{2}\binom{3}{1} & \cdots & \alpha^{1-n}\binom{n}{1} \\ 0 & 0 & \alpha^0\binom{2}{2} & \alpha^{-1}\binom{3}{2} & \cdots & \alpha^{2-n}\binom{n}{2} \\ 0 & 0 & 0 & \alpha^0\binom{3}{3} & \cdots & \alpha^{3-n}\binom{n}{3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha^0\binom{n}{n} \end{pmatrix}. \tag{12}$$

Let

$$\mathbf{P} := \mathbf{P}(-1) = \begin{pmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \binom{3}{0} & \cdots & \binom{n}{0} \\ 0 & \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \cdots & \binom{n}{1} \\ 0 & 0 & \binom{2}{2} & \binom{3}{2} & \cdots & \binom{n}{2} \\ 0 & 0 & 0 & \binom{3}{3} & \cdots & \binom{n}{3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \binom{n}{n} \end{pmatrix} \tag{13}$$

and

$$\mathbf{A}(-\alpha) := \begin{pmatrix} \alpha^{0} & \alpha^{-1} & \alpha^{-2} & \alpha^{-3} & \cdots & \alpha^{-n} \\ 0 & \alpha^{0} & \alpha^{-1} & \alpha^{-2} & \cdots & \alpha^{1-n} \\ 0 & 0 & \alpha^{0} & \alpha^{-1} & \cdots & \alpha^{2-n} \\ 0 & 0 & 0 & \alpha^{0} & \cdots & \alpha^{3-n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha^{0} \end{pmatrix}. \tag{14}$$

Then,

$$\mathbf{P}(\alpha) = \mathbf{A}(\alpha) \star \mathbf{P} = \mathbf{P} \star \mathbf{A}(\alpha), \tag{15}$$

where $\star$ denotes the Hadamard product. Since $\mathbf{P}^{-1}(\alpha) = \mathbf{P}(-\alpha)$ we obtain

$$\mathbf{P}^{-1}(\alpha) = \mathbf{A}(-\alpha) \star \mathbf{P}. \tag{16}$$

Next, suppose that

$$c := \sum_{i=1}^{m} \alpha_i. \tag{17}$$

Then,

$$c = \sum_{i=1}^{m} \pi(\alpha_i), \tag{18}$$

where π denotes any permutation of $\{1, \ldots, m\}$. Hence,

$$\mathbf{P}(c) = \prod_{i=1}^{m} \mathbf{P}(\alpha_i) = \prod_{i=1}^{m} \mathbf{P}(\pi(\alpha_i)). \tag{19}$$

Notice that the set of matrices $\{\mathbf{P}(\alpha), \alpha \in \mathbb{R}\}$ form a multiplicative abelian group with inverse $\mathbf{P}^{-1}(\alpha) = \mathbf{P}(-\alpha)$ and identity matrix $\mathbf{P}(0)$.

IV. Properties of the Transformation Matrices Associated with Multivariable Polynomials

This Section is based on the works in (Rao and Aatre, 1976) and (Hertz and Zeheb, 1987). The transformation

$$u_i := v_i/\beta_i - \alpha_i,\ \beta_i \neq 0,\ i = 1, \ldots, m \tag{20}$$

of a multi-variable polynomial

$$f(\mathbf{u}) := f(u_1, \ldots, u_m) \tag{21}$$

is useful in various engineering applications, *e.g.*, design and stability tests of multidimensional filters. We define the transformed polynomial by

$$F(\mathbf{v}) := f(v_1/\beta_1 - \alpha_1, \ldots, v_m/\beta_m - \alpha_m). \tag{22}$$

Let N_i denotes the highest degree of u_i in $f(\mathbf{u})$ which also turns out to be the highest degree of v_i in $F(\mathbf{v})$. Following the notation in (Rao and Aatre, 1976), let $\tilde{\mathbf{a}}$ denote the vector of coefficients of $f(\mathbf{u})$ arranged in such a way that the coefficient of the monomial $\Pi_{i=1}^{m} u_i^{\gamma_i}$, $(0 \leq \gamma_i \leq N_i,\ i = 1, \ldots, m)$ is the k'th component of $\tilde{\mathbf{a}}$, where

$$k = (N_1 - \gamma_1)(N_2 + 1) \cdots (N_m + 1) + (N_2 - \gamma_2)(N_3 + 1) \cdots (N_m + 1) + \cdots + (N_m - \gamma_m) + 1. \tag{23}$$

A similar notation holds for the vector $\tilde{\mathbf{b}}$ of the coefficients of $F(\mathbf{v})$. Evidently, the number of coefficients of $\tilde{\mathbf{a}}$ (and also of $\tilde{\mathbf{b}}$) including those with zero value is $N := \Pi_{i=1}^{m}(N_i + 1)$. Finally, the relation between $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}}$

$$\tilde{\mathbf{b}} := \tilde{\mathbf{M}}\tilde{\mathbf{a}} \tag{24}$$

defines the transformation matrix $\tilde{\mathbf{M}}$ of order $N \times N$. The multlivariable transformation matrix $\tilde{\mathbf{M}}$ as constructed above can be similarly obtained as in (Rao and Aatre, 1976) , *i.e.*,

$$\tilde{\mathbf{M}} = \tilde{\mathbf{P}}_1(\alpha_1, \beta_1) \otimes \cdots \otimes \tilde{\mathbf{P}}_m(\alpha_m, \beta_m), \tag{25}$$

where $\otimes$ denotes the Kronecker product and $\tilde{\mathbf{P}}_i(\alpha_i, \beta_i)$, denotes the $(N_i + 1)$-square transformation matrix that transforms the vector $\tilde{\mathbf{p}} := (p_{N_i}, p_{N_i-1}, \ldots, p_0)^T$ corresponding to $f_i(u_i) = \Sigma_{k=0}^{N_i} p_k u_i^k$ in the variable u_i to the vector $\tilde{\mathbf{q}} := (q_{N_i}, q_{N_i-1}, \ldots, q_0)^T$ corresponding to $F_i(v_i) = f_i(v_i/\beta_i - \alpha_i) = \Sigma_{k=0}^{N_i} q_k v_i^k$ in the variable v_i. For a comprehensive treatment of the Kronecker product see (Marcus 1993).

When $m = 1$ the vector $\tilde{\mathbf{a}} = (a_{N_1}, \ldots, a_0)^T$ can be obtained from the vector of coefficients $\mathbf{a} = (a_0, \ldots, a_{N_1})^T$ by flipping its elements. *i.e.*, $\tilde{\mathbf{a}} = \mathbf{J}_1\mathbf{a}$, where $\mathbf{J}_1$ denotes the $(N_1 + 1)$-permutation matrix whose secondary diagonal elements are all ones. Since $\mathbf{J}_1^2 = \mathbf{I}$, $\mathbf{J}_1$ is involutory and $\mathbf{J}_1^{-1} = \mathbf{J}_1$. Involutory matrices are defined as the square roots of the identity matrix and $\mathbf{J}_1$ is one of them.

Hence, for $m = 1$ and $\tilde{\mathbf{a}} = \mathbf{J}_1\mathbf{a}$ we obtain

$$\begin{aligned}\tilde{\mathbf{b}} &= \mathbf{J}_1\mathbf{b} \\ &= \mathbf{J}_1[\mathbf{P}_1(\alpha, \beta)\mathbf{a}] \\ &= \mathbf{J}_1\mathbf{P}_1(\alpha, \beta)\mathbf{J}_1\mathbf{J}_1\mathbf{a} \\ &= [\mathbf{J}_1\mathbf{P}_1(\alpha, \beta)\mathbf{J}_1]\tilde{\mathbf{a}}.\end{aligned} \tag{26}$$

Therefore,

$$\tilde{\mathbf{P}}_1(\alpha, \beta) = \mathbf{J}_1\mathbf{P}_1(\alpha, \beta)\mathbf{J}_1. \tag{27}$$

Similarly, as in the previous section we let $\tilde{\mathbf{P}}_i(\alpha) := \tilde{\mathbf{P}}_i(\alpha, 1)$ and $\tilde{\mathbf{P}}_i := \tilde{\mathbf{P}}_i(-1)$. Notice that Pascal's triangle appears in $\tilde{\mathbf{P}}_i$ in its lower right corner.

Using the relationship $(\mathbf{A}_1 \otimes \mathbf{B}_1)(\mathbf{A}_2 \otimes \mathbf{B}_2) = (\mathbf{A}_1\mathbf{A}_2) \otimes (\mathbf{B}_1\mathbf{B}_2)$ where we assume that $\mathbf{A}_1\mathbf{A}_2$ and $\mathbf{B}_1\mathbf{B}_2$ exist we obtain (Marcus 1993)

$$\begin{aligned}\tilde{\mathbf{M}} &= \tilde{\mathbf{P}}_1(\alpha_1, \beta_1) \otimes \cdots \otimes \tilde{\mathbf{P}}_m(\alpha_m, \beta_m) \\ &= [\mathbf{J}_1\mathbf{P}_1(\alpha_1, \beta_1)\mathbf{J}_1] \otimes \ldots \otimes [\mathbf{J}_m\mathbf{P}_m(\alpha_m, \beta_m)\mathbf{J}_m] \\ &= \mathbf{J} \otimes [\mathbf{P}_1(\alpha_1, \beta_1) \otimes \ldots \otimes \mathbf{P}_m(\alpha_m, \beta_m)] \otimes \mathbf{J},\end{aligned} \tag{28}$$

where

$$\mathbf{J} := \mathbf{J}_1 \otimes \ldots \otimes \mathbf{J}_m \tag{29}$$

is the $N := \Pi_{i=1}^{n}(N_i + 1)$ square involutory matrix similar to the $\mathbf{J}_i$'s.

Now, let

$$\mathbf{M} = \mathbf{P}_1(\alpha_1, \beta_1) \otimes \ldots \otimes \mathbf{P}_m(\alpha_m, \beta_m). \tag{30}$$

Then,

$$\begin{aligned}\mathbf{J}\tilde{\mathbf{b}} &= \mathbf{J}\tilde{\mathbf{M}}\tilde{\mathbf{a}} \\ &= \mathbf{JJMJ}\tilde{\mathbf{a}} \\ &= \mathbf{MJ}\tilde{\mathbf{a}}.\end{aligned} \tag{31}$$

Hence, if we let $\mathbf{a} := \mathbf{J}\tilde{\mathbf{a}}$ and $\mathbf{b} := \mathbf{J}\tilde{\mathbf{b}}$ be the flipped versions of $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ we obtain

$$\mathbf{b} = \mathbf{Ma}. \tag{32}$$

Next, by using the relationship $(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$ we obtain (Marcus 1993)

$$\begin{aligned}\mathbf{M}^{-1} &= \mathbf{P}_1^{-1}(\alpha_1, \beta_1) \otimes \cdots \otimes \mathbf{P}_m^{-1}(\alpha_m, \beta_m) \\ &= \mathbf{P}_1(-\alpha_1\beta_1, \beta_1) \otimes \cdots \otimes \mathbf{P}_m(-\alpha_m\otimes_m, \beta_m^{-1}) \\ &= [\mathbf{P}_1(-\alpha_1)\mathbf{D}_1(\beta_1^{-1})] \otimes \cdots \otimes [\mathbf{P}_m(-\alpha_m)\mathbf{D}_m(\beta_m^{-1})]\end{aligned} \tag{33}$$

When $\beta_i = 1$, $\alpha_i = \Sigma_{j=1}^{\ell}\alpha_{i,j}$ $(i = 1, ..., m)$, and $\pi_i(i = 1, ..., m)$ are arbitrary permutations of $\{1, ..., \ell\}$ we have

$$\begin{aligned}\mathbf{M} &= \mathbf{P}_1(\alpha_1) \otimes \cdots \otimes \mathbf{P}_m(\alpha_m) \qquad (34)\\ &= \mathbf{P}_1\left(\sum_{j=1}^{\ell}\alpha_{1,j}\right) \otimes \cdots \otimes \mathbf{P}_m\left(\sum_{j=1}^{\ell}\alpha_{m,j}\right)\\ &= \mathbf{P}_1\left(\sum_{j=1}^{\ell}\alpha_{1,\pi_1(j)}\right) \otimes \cdots \otimes \mathbf{P}_m\left(\sum_{j=1}^{\ell}\alpha_{m,\pi_m(j)}\right)\\ &= \left[\Pi_{j=1}^{\ell}\mathbf{P}_1(\alpha_{1,\pi_1(j)})\right] \otimes \cdots \otimes \left[\Pi_{j=1}^{\ell}\mathbf{P}_m(\alpha_{m,\pi_m(j)})\right]\\ &= \Pi_{j=1}^{\ell}[\mathbf{P}_1(\alpha_{1,\pi_1(j)}) \otimes \cdots \otimes \mathbf{P}_m(\alpha_{m,\pi_m(j)})].\end{aligned}$$

Hence, the set of matrices defined in (34), *i.e.*, $\{\mathbf{M}(\alpha_1, ..., \alpha_m) : (\alpha_1, ..., \alpha_m) \in \mathbb{R}^m\}$ is commutative with respect to multiplication with an additional inner level of commutativity and form a multiplicative abelian group with identity matrix $\mathbf{I} = \mathbf{M}\underbrace{(0, ..., 0)}_{m}$.

V. Conclusion

In this article we presented a simple upper bound on the stability margin of a given a Hurwitz polynomial, say $f(s) := \Sigma_{i=0}^{n} a_i s^i$. The proposed upper bound is $\sigma_u := a_{n-1}/n$, where a_{n-1} is $f(s)$'s coefficient of s^{n-1}. This upper bound coincides with the α that renders $F(S) = f(S - \alpha)$ a depressed polynomial, *i.e.*, where the coefficient of S^{n-1} becomes zero. This bound can be used to narrow the search region for the stability margin σ of $f(s)$ from (σ_ℓ, ∞) to (σ_ℓ, σ_u), where $\sigma_\ell \geq 0$ is an initial lower bound on the stability margin.

Next, we showed that the matrix $\mathbf{P}(\alpha, \beta)$ that transforms the coefficients of $f(s)$ to the coefficients of $F(S) := f(S/\beta - \alpha)$, $\beta \neq 0$ satisfies $\mathbf{P}^{-1}(\alpha, \beta) = \mathbf{P}(-\alpha\beta, \beta^{-1})$. We defined $\mathbf{P}(\alpha) := \mathbf{P}(\alpha, 1)$, and $\mathbf{P} := \mathbf{P}(-1)$ and called the upper triangular matrix $\mathbf{P}$ Pascal's matrix because its nonzero elements coincide with Pascal's triangle. We have shown the following results. By shifting the roots of $f(s)$ by α and then shifting the roots of $F(S)$ by $-\alpha$ we arrive back at $f(s)$. Hence, we obtain that $\mathbf{P}(-\alpha)\mathbf{P}(\alpha) = \mathrm{I}$, where I denotes the $(n + 1)$-square identity matrix and consequently $\mathbf{P}^{-1}(\alpha) = \mathbf{P}(-\alpha)$ thus generalizing Waring's formulas, *i.e.*, $\mathbf{P}^{-1} = \mathbf{P}(1)$. Similarly, we derived other matrix identities where multiplication is commutative. We could thus show that the set of matrices $\{\mathbf{P}(\alpha) : \alpha \in \mathbb{R}\}$ form a multiplicative abelian group with identity matrix $\mathbf{P}(0)$. We have also shown that $\mathbf{P}(\alpha, \beta) = \mathbf{D}(\beta)\mathbf{P}(\alpha)$, where $\mathbf{D}(\beta)$ is a diagonal matrix whose elements are $\mathbf{D}_{ii}(\beta) = \beta^{-i}$, $i = 0, 1, ..., n$. Therefore, $\mathbf{P}^{-1}(\alpha, \beta) = \mathbf{P}(\alpha)^{-1}\,\mathbf{D}^{-1}(\beta) = \mathbf{P}(-\alpha)\mathbf{D}(\beta^{-1})$. Finally, we extended the above results to multivariable polynomials with similar conclusions and with richer structure. Further research will focus on finding other operations on polynomials that can be translated to matrix operations and eventually to fast algorithms. For some more examples of this line of research refer to (Hertz, 1991), (Hertz, 2021), and (Hertz, 2022).

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